BACK

ARITHMETIC OPERATIONS WITH POLYNOMIALS ON TI-92

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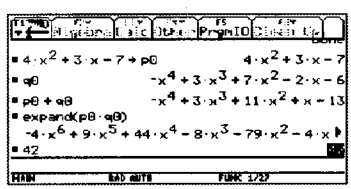
Abstract
Giving for many years the courses of linear and higher algebra to our students -future secondary school teachers, we found out that they usually do not see the important connections between different algebraical structures they are taught. For example, working with polynomials in one variable they do not use the advantage of the fact that this is a linear space, though they have previously studied in details the structure of a vector space. Modern computer algebra systems (CAS) offer new possibilities for establishing various links between different mathematical structures and we believe that this is one of the ways to consolidate the fragmented students knowledge. As a concrete example, in this paper we are dealing with vector notation of polynomials on the TI-92 calculators. We are also demonstrating how some of the most common operations on polynomials can be performed.

Given an arbitrary ring R (for instance the set of all integers $\mathbb{Z}$) and an abstract variable x we define a polynomial p in variable x over the ring R as a formal sum of the form $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$. On the set $R[x]$ of all such polynomials the operations of addition and multiplication are introduced in the following way. For any two polynomials $p = \sum_{i=0}^{n} a_i x^i$ and $q = \sum_{i=0}^{m} b_i x^i$ we put $p+q = \sum_{i=0}^{n}(a_i + b_i)x^i$ where in case of $n > m$ we assume $b_i = 0$ for all $i > m$ (and vice versa), and $pq = \sum_{k=0}^{n+m} c_k x^k$ with $c_k = \sum_{i=0}^{k} a_i b_{k-i}$.

It is an easy exercise to check that in this way $R[x]$ itself becomes a ring, and even an algebra if the original ring R is assumed to be a field. The CAS of TI-92 includes the standard polynomial arithmetics (for polynomials of one and several variables).



While the operation of polynomial addition is quite transparent and consists simply of a certain number ($\min\{m, n\}+1$) of additions in the main ring R, the rule for the polynomial multiplication is less obvious and performing it on TI-92 in a "black-box"' manner students hardly get an impression of its complexity. Only looking carefully what happens with the coefficients of both polynomials during their multiplication; the students realize that in this case the calculator has to perform $(m+1)(n+1)$ multiplications and mn additions of the coefficients of polynomials p and q.

Students in Slovenia are, in accordance with our mathematical curriculum and through physics, quite early introduced to the concept of vectors in a standard linear space $\mathbf{R}^n$ with the operations of vector addition, multiplication by scalars and the dot product. The parallel between the first two operations for polynomials and vectors is quite obvious. To examine the operation of multiplication in $R[x]$ let us first find a convenient notation for polynomials in a vector form. One of the possibilities is to connect the polynomial $p = \sum_{i=0}^{n} a_e x_i$ $v_p=(a_0, a_1, a_2, \ldots a_n)$. It is not difficult for the students to find the pattern for this conversion on TI-92. On one hand, starting with the polynomial $p = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_0$ it is sufficient to evaluate at point 0 the sequence of $n+1$ polynomials obtained by the recursive formula: $p_0 = p$ and $p_i = \dfrac{p_{i-1} - p_{i-1}(0)}{x}$ for $1 \leq i \leq n$. Then $v_p = (p_0(0),\ p_1(0), \ldots\ p_n(0))$.

Similarly, starting with an arbitrary vector $v_p=(a_0, a_1, a_2, \ldots a_n) \in R^n$ we can reconstruct the corresponding polynomial p by the following steps:

$p_0 = a\_n$

$p_1 = x \cdot p_0 + a_{n-1}$

$\vdots$

$p_n = x \cdot p_{n-1} + a_0 = \sum_{i=0}^{n} a_i x^i = p$

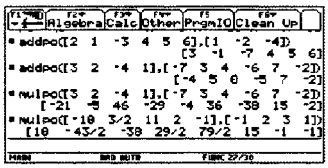
The conversion functions on TI-92 may look like

```
pove(pp,xx)
Func
Local aa,ii,qq,vv
pp|xx=0→aa
pp-aa→qq
[[aa]]→vv
While (qq|xx=$\pi)\not=0$
qq/xx→qq
qq|xx=0→aa
qq-aa$→qq
augment(vv,[[aa]])→vv
EndWhile
EndFunc
```

```
vepo(vv,xx)
Func © Convert a vect to a polynom
Local ii,jj,mm
dim(mat▷list(vv))→ii
dotP([[1]],subMat(vv,1,ii,1,ii))→mm
For jj,ii-1,1,-1
mm*xx+km(vv,jj)→mm
EndFor
expand(mm)
EndFunc
```

As the size of the obtained vector v_p is equal to the degree of the related polynomial p increased by 1, the students are not able to use directly the + command for the pair of vectors (v_p, v_q) if polynomials p and q have different degrees. It is first necessary to complete the shorter of the two vectors by the appropriate number of zeros.

In a similar way we can program TI-92 to perform the multiplication of two polynomials, written in the above vector form:



In this way we need m+1 multiplications of vectors, whose size varies from n+1 to n+m+1, by scalars and m additions of the resulting vectors.

Many times we are interested in the value $p(x_0)$ of a polynomial $\sum_{i=0}^{n} a_i x^i$ at the specific point $x_0 \in R$. It is well known that the optimal way to find this value by just n multiplications and n additions in the underlying ring R is the so called Horner algorithm. In this way we obtain the coefficients of the quotient $q(x) = \sum_{i=0}^{n-1} c_i x^i$ of the division of polynomial p by binomial $x - x_0$.

As $c_i = \begin{cases} a_n, & \text{if } i = n-1 \\ a_{i+1} + c_{i+1}b, & \text{for } i < n-1 \end{cases}$ and $p(b)=a_0+c_0b$ the Horner algorithm can be implemented on TI-92 by using a single loop:

```
hornal(vv,pp)
Func
Local ii,kk,rr
dim(mat▷$\triangleright$list((vv))→kk
subMat(vv,1,kk,1,kk) →rr
For ii,kk-1,1,-1
augment(rr,subMat(vv,1,ii,1,ii)+pp*subMat(rr,1,kk-ii,1,kk-ii))
→
EndFor
EndFunc
```

For instance, the hornal([3,2,-4,5,1],-2) results in [1,3,-10,22,-41], therefore the value of the polynomial $x^4+5x^3-4x^2+2x+3$ at $x = -2$ is -41.

Every polynomial of degree n is uniquely determined by its values on a set containing $n+1$ points, $S=\{x_0, x_1, \ldots x_n\}$. Thus we have another possibility to adjoin a vector $w_{p,S}=(p(x_0), p(x_1), \ldots p(x_n))$ to every polynomial p. This mapping is 1-1 and obviously $w_{p,S} + w_{q,S} = w_{p+q,S}$, $w_{p,S} \odot w_{q,S} = w_{p\,.\,q,S}$.

To reconstruct the sum and the product of the original polynomials we must include in the set S at least max{deg p, deg q}+1 points in the first case and at least deg p + deg q + 1 points in the second case.

By $\odot$ we have denoted the ``point by point" multiplication of two vectors, for which on the TI-92 calculator the symbol * is used. The transformation of a polynomial p into the vector $w_{p,S}$ is easy by the **mat▷list** and **polyEval** commands from the TI-92 library.

References

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- Graham R. L., Knuth D. E., Patashnik O.: *Concrete Mathematics*, Addison-Wesley 1994.