BACK

LINEAR ALGEBRA WITH MAPLE

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Abstract:
Since Fall 1999, in Quebec, the Science Program of Cegeps "Collèges d'Enseignement Général et Professionels", where students come after 5 years of high school, was reformed in order to ensure that science students will become computer-literate. As an example, for the two courses of Mathematics: Integral Calculus and Linear Algebra, it was advised to introduce the students to the Canadian CAS (Computer Algebraic System) Maple. We would like to illustrate the benefits brought by Maple in the teaching of Linear Algebra. For this discipline, students are asked to work in successive frameworks: Vector spaces, Systems of linear equations, Algebra of matrices, determinants and linear transformations; we shall show a pedagogical scenario dealing with the parameter time, that should succeed to exhibit unity in Linear Algebra representations.

Linear dependence and coplanarity.

Under which conditions is a moving vector depending of the time, a linear combination of two given vectors. How can we check it? We can invoke the resolution of a system of linear equations, evaluate a determinant or check geometrically when the vector is lying nicely in the plane spanned by the two initial vectors. We can use then different frameworks for the same question. We keep on purpose Maple notations in what will follow.

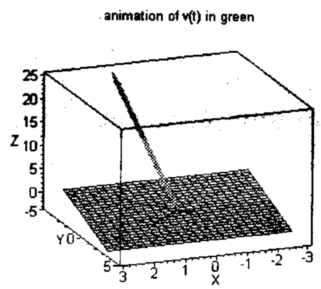
Let two vectors u1 = [1,-1,1] and v1 = [1,1,-1]; the moving vector is v = [1,t,t^2;]. When is v a linear combination of u1 and u2, i.e. v = c1*u1+c2*u2? We need to solve a system of 3 linear equations with the 2 unknowns c1 and c2. We may see that the augmented matrix A(t) will be composed of the 3 columns equal to the 3 vectors in this order, u1, u2 and v. In general the system is inconsistent. It means that in general v is not a linear combination of u1and u2 and so the vector v is not in the plane spanned by u1 and u2. Critical values of t for which the vector v is in the plane spanned by u1 and u2 can be found when the triple product of the 3 vectors is 0 or when the determinant of A(t) is zero. It will happen for t= 0 or –1.

To run the animation of this first set-up, we calculate the equation of the plane P spanned by u1 and u2 using crossprod(u1,u2) = [0, 2, 2]. Then the equation of this plane that goes through the origin (it is a subspace) is 2 y + 2 z = 0 or simplified to y+z =0;

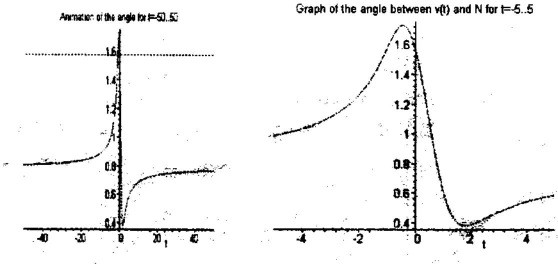
Here we give the paragraph of Maple commands to run the animation, as an example of Maple syntaxes. We will not repeat it.

```
> plot3d(-y,x=-5..5,y=-5..5,axes = boxed, title=`the plane y + z = 0`);
> o:=[0,0,0]:
> U2:=arrow(o,u2,.5,1,.5,color=blue):
> NN:=arrow(o,N,.2,.6,.2,color=black):
> NN10:=arrow(o, N10,.5,1,.5,color=black):
> U1:=arrow(o,u1,.5,1,.5,color=red):
> P:=plot3d(-y,x=-3..3,y=-3..3,axes = boxed,color=yellow):
> V:=display(seq(arrow(o,v,.5,.5,.1),t=-5..5),insequence=true,frames=22,color=green):
> display(U1,U2,P,NN,axes=boxed, labels=[X,Y,Z],orientation=[72,58],title=`Plane spanned by the blue and red vectors u1 and u2, normal to N`);
>display(U1,U2,P,NN,V,axes=boxed,
labels=[X,Y,Z],orientation=[74,69],frames=22,title=`animation of v(t) in green`);
>
```

Here is one frame of the animation:



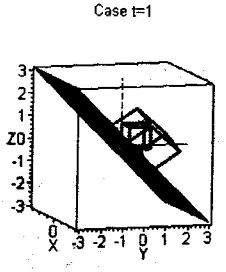
We may question the students about the outcome of the vector v to be normal to the plane P; does it occur? We should find the value of t when we have the equality v(t) = cN. Here are some graphs after few Maple calculations.

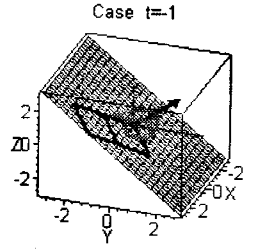


We can understand from this graph what we should have seen during the animation: the vector v (t) is in the plane when the angle with the normal is Pi/2 for the instants t= -1 and t = 0; the smallest angle equal to 0.38 radian seems to occur for t=1.82; this is close to 22 degrees, or an angle of 90 - 22 = 68 degrees, the closest to the normal. For the long term behaviour, we may calculate the limit and trace the horizontal asymptote at Pi/4.

Linear transformation, rank, range and kernel.

Now we are going to use again the 3 x 3-matrix A(t); it will be the code of a linear transformation F(t) of R^3. Observe the various images of the unit box built on the 3 vectors i, j and k. The students should realize that we have a strong relation with the first problem: F(i) = u1 , F(j) = u2 and F(k) = v(t) . We may during the animation identify the range and the kernel (nullspace) of the linear transformation at that instant. We have always one base of the image box in the plane P as the images of the two unit vectors i and j of the initial unit box are u1 and v1. The unit volume will be multiplied by det(A(t)). For example at t=1, the new box has a volume = 4. For t=0 and –1, the matrix A(t) is now of determinant = 0. As the matrix is not invertible, the volume of the image box should be 0; it means that we expect the new box to collapse onto the plane P. The ranges of F(0) and F(-1) are equal to the plane P. The ranks are 2, so the nullspace (kernel) should be of dimension 3-2 = 1. The nullspaces of F(-1) and F(0) are respectively the lines spanned respectively by [-1,0,1] and [1,1,-2]. It can be easily calculated and then shown during a Maple animation.





Equation of Planes, Rowspaces and Nullspaces

Now we are going to consider the homogenous system of equations:

A*X = O, compare the solutions space (the nullspace of A with the three planes described by the 3 rows). The system is the following:

$$\begin{aligned} x + y + z &= 0 \\ -x + y + tz &= 0 \\ x + -y + t^2 &= 0 \end{aligned}$$

We have then one fixed plane with normal vector [1,1,1] and two moving planes. All of them contain the origin as we have a homogenous system;

At what time is the intersection of the three planes equal only to the origin and when do we get a larger intersection? Of course this problem is strongly related to the previous problems in A) and B).